CERTAIN NONISOTHERMAL FLOWS OF FLUID

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1. It is known that the absolute viscosity of fluids μ depends to a significant extent on temperature (the greater T, the lower μ), while other parameters change little with temperature.

The exponential relation below [1] is usually used for very viscous fluids (such as glycerin)

$$\mu = \mu_0 \exp\{-\beta(T - T_0)\} \tag{1.1}$$

(μ_0, β, and T_0 are empirical constants). For ordinary fluids (such as water), the curve of the dependence of viscosity on temperature is more shallow and can in many cases be described by the power function [2]

$$\mu = \mu_0(T_0/T)^m, \ m \geqslant 0. \tag{1.2}$$

In certain temperature ranges and with the appropriate values of the exponent m, Eq. (1.2) is a good approximation of the well-known function of J. I. Frenkel [3].

It should be noted that the studies [4-6] examined nonisothermal rectilinear flows in pipes with allowance for dissipative heating and the temperature dependence of viscosity. In these problems, there was no convective heat transfer or temperature gradient along the walls. The authors of [7, 8] studied certain hydrodynamic problems with a temperature gradient along the pipe wall, when convection plays an important role. Here, it was assumed that viscosity depended exponentially on temperature.

In the present study, we generalize the results from [7, 8] in several directions: first, we examine both Eqs. (1.1) and (1.2); secondly, we study non-Newtonian fluids whose apparent viscosity depends arbitrarily on the quadratic invariant of the strain-rate tensor; thirdly, we examine a model problem of nonisothermal flow in a porous medium to analyze the stability of the steady-state solution we obtain.

2. Movement of Fluid in a Porous Medium. The simplest model of the slow nonisothermal flow of an incompressible fluid in a homogeneous porous medium is described by the equation

$$\mathbf{v} = -\frac{k}{\mu(T)}\nabla p, \mathbf{v} = (v_1, v_2, v_3); \tag{2.1}$$

$$\operatorname{div}\mathbf{v} = 0; \tag{2.2}$$

$$\partial T/\partial t + (\mathbf{v}\nabla)T = \varkappa\Delta T, \tag{2.3}$$

where v_1, v_2, and v_3 are the components of fluid velocity in a Cartesian coordinate system x, y, z; p is pressure; $\varkappa$ is thermal diffusivity; k is the permeability of the soil; t is time.

The nonsteady terms have been omitted from Eqs. (2.1). This approximation is valid for most real systems [9, 10].

We will examine a rectilinear

$$v_1 = 0, \ v_2 = 0, \ v_3 = w(t, x, y) \tag{2.4}$$

flow in a pipe whose wall is kept at a temperature that changes exponentially with the longitudinal coordinate z:

$$T|_S = T_0 e^{-\lambda} \tag{2.5}$$

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(S is the contour of the cross section of the pipe). Assuming that viscosity depends exponentially on temperature (1.2) and allowing for (2.4)-(2.5), we seek particular solutions of system (2.1)-(2.3) in the form

$$T = T_0 e^{-\lambda z} f,\ w = \frac{k}{\mu_0}\frac{p_*}{h} f^m,\ \frac{dp}{dz} = -\frac{p_*}{h} e^{\lambda m z}. \tag{2.6}$$

Here, h is the characteristic dimension of the cross section of the pipe; p_* is a constant chosen as the pressure scale. Inserting Eqs. (2.6) into (2.3) and (2.5) [Eqs. (2.1)-(2.2) are satisfied automatically by virtue of (2.4), (2.6)], we obtain the equation and boundary condition for the unknown function f:

$$\partial f/\partial\tau = \partial^2 f/\partial\xi^2 + \partial^2 f/\partial\eta^2 + \Lambda^2 f + P f^{m+1}; \tag{2.7}$$

$$f|_S = 1 \tag{2.8}$$

$$(\tau = \varkappa t/h^2, \xi = x/h, \eta = y/h, \Lambda = \lambda h, P = k\lambda h p_*/(\mu_0\varkappa)).$$

We will study a flow in a plane channel of the width 2h ($-h \le x \le h$). The flow is symmetrical relative to the midline x = 0. In this case, $\partial^2/\partial\eta^2 = 0$ and it is sufficient to examine the region $0 \le \xi \le 1$. Considering the foregoing, we find from (2.7)-(2.8) that

$$\partial f/\partial\tau = \partial^2 f/\partial\xi^2 + \Lambda^2 f + P f^{m+1}; \tag{2.9}$$

$$\xi = 0{:}\ \partial f/\partial\xi = 0;\ \xi = 1{:}\ f = 1. \tag{2.10}$$

The steady motion of the flow is described by an ordinary differential equation and boundary conditions

$$\bar f''_{\xi\xi} + \Lambda^2\bar f + P\bar f^{m+1} = 0,\ \bar f'_\xi(0) = 0,\ \bar f(1) = 1. \tag{2.11}$$

The exact solution of problem (2.11) can be written in implicit form

$$\int\limits_{1/f_0}^{\bar f/f_0}\left[\frac{2P}{m+2} f_0^m\left(1 - u^{m+2}\right) + \Lambda^2\left(1 - u^2\right)\right]^{-1/2} du = 1 - \xi, \tag{2.12}$$

where $f_0 = \bar f(0)$ is the temperature on the axis of the channel. This temperature is determined from the solution of the transcendental equation

$$\int\limits_{1/f_0}^{1}\left[\frac{2P}{m+2} f_0^m\left(1 - u^{m+2}\right) + \Lambda^2\left(1 - u^2\right)\right]^{-1/2} du = 1. \tag{2.13}$$

It follows from the second relation of (2.6) that f_0 depends exponentially on the maximum flow velocity: $f_0 \sim [w(0)]^{1/m}$.

At m = 2, Eq. (2.12) can be written in terms of elliptic integrals in the form:

$$F(\varphi_0, f_0/c) - F(\varphi, f_0/c) = c(P/2)^{1/2}(1-\xi).$$

Here, $\cos\varphi_0 = 1/f_0$; $\cos\varphi = 1/f$; $c = \sqrt{1 + f_0^2 + 2\Lambda^2 P^{-1}}$; F is an incomplete elliptic integral of the first kind [11]. With assigned P and Λ, we find the parameter $f_0 = f_0(P, \Lambda)$ from the transcendental equation $F(\varphi_0, f_0/c) = c(P/2)^{1/2}$.

Fixing the parameter $\Lambda \ge 0$, we will examine qualitative features of the dependence of temperature on the axis of the channel f_0 on the dimensionless pressure P. It follows from Eq. (2.13) that $f_0 \to 1$ corresponds to $P \to 0$. On the other hand, $P \to 0$ at $f_0 \to \infty$ as well. Thus, there exists a value P_{max} such that nonlinear boundary-value problem (2.11) has no solution at $P > P_{max}$, while at $0 < P < P_{max}$ each $P \in (0, P_{max})$ corresponds to two solutions with different temperatures on the channel axis. On one branch of the solution, $dP/df_0 < 0$ (the pressure gradient decreases with an increase in flow velocity), while $dP/df_0 > 0$ on the other branch (the pressure gradient increases with fluid velocity). Figure 1 shows the relation $P(f_0)$ at $\Lambda = 0$, which corresponds to the assumption that axial heat conduction is small compared to convective heat transfer (here and in Figs. 2 and 3, the dashed line corresponds to unstable regimes).

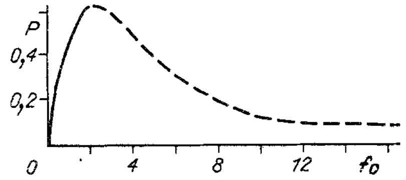

Fig. 1

We will show that the steady-state solution is unstable at $dP/df_0 < 0$. Let $f = \bar{f} + \varepsilon$ ($\bar{f}$ is the steady-state solution of (2.11), while $\varepsilon = \varepsilon(\tau, \xi)$ are the perturbed solutions). At small ε, we obtain the following if we linearize (2.9) in the neighborhood of the steady-state solution

$$\partial\varepsilon/\partial\tau = \partial^2\varepsilon/\partial\xi^2 + \Lambda^2\varepsilon + (m+1)P\bar{f}^m\varepsilon. \tag{2.14}$$

We seek the solution of Eq. (2.14) in the form

$$\varepsilon = \omega(\xi)\, e^{\beta_n \tau}. \tag{2.15}$$

Inserting (2.15) into (2.14) with allowance for (2.10), we have the following spectral problem to determine the exponent β_n

$$\omega''_{\xi\xi} + \left[(m+1)P\bar{f}^m + \Lambda^2 - \beta_n\right]\omega = 0,\ \omega'_\xi(0) = 0,\ \omega(1) = 0. \tag{2.16}$$

The spectral properties of Eq. (2.16) are well known [12, 13]. There is a discrete spectrum of eigenvalues; here, the eigenfunction corresponding to the lowest eigenvalue β_0 does not change sign on the interval (0, 1).

Differentiating Eq. (2.11) with respect to f_0 and designating $\psi = d\bar{f}/df_0$, we write

$$\psi''_{\xi\xi} + \Lambda^2\psi = -(m+1)P\bar{f}^m\psi - \frac{dP}{df_0}\bar{f}^{m+1}. \tag{2.17}$$

We multiply Eq. (2.17) by the sign-constant eigenfunction ω_0 of problem (2.16). We then integrate the resulting expression over ξ from 0 to 1 with allowance for the equalities

$$\omega_0\psi''_{\xi\xi} = (\omega_0\psi'_\xi - \omega'_{0\xi})'_\xi + \omega''_{0\xi\xi}\psi = (\omega_0\psi'_\xi - \omega'_{0\xi}\psi)'_\xi + \left[\beta_0 - \Lambda^2 - (m+1)P\bar{f}^m\right]\omega_0\psi.$$

As a result, we arrive at the relation

$$\beta_0\int_0^1 \omega_0\psi\, d\xi = -\frac{dP}{df_0}\int_0^1 \bar{f}^{m+1}\omega_0\, d\xi. \tag{2.18}$$

In deriving (2.18), we used the boundary conditions for the function ω (2.16) and the function ψ: $\psi_\xi'(0) = 0$, $\psi(1) = 0$.

Since the sign of ω_0 remains constant at $0 \le \xi \le 1$, $\bar{f} \ge 0$, and $\psi \ge 0$ (the temperature of the fluid increases with an increase in the temperature on the channel axis), then the sign of β_0 is opposite the sign of the derivative dP/df_0. Thus, when $dP/df_0 < 0$, we have $\beta_0 > 0$, i.e., the corresponding solution is unstable. This is what we had to prove. At $dP/df_0 > 0$, $\beta_0 < 0$, and the steady-state solution is stable against small perturbations.

We will similarly examine a more general case, when the permeability of the ground depends on pressure $k = k(p, |\nabla p|)$. We seek the particular solution for temperature T in the form (2.6), while the equation for the function f will again be (2.7). It should be noted that the authors of [8] studied steady nonisothermal motion in a porous medium for a linear change in wall temperature with an exponential temperature dependence of viscosity.

3. Non-Newtonian Fluid. We will study nonisothermal flows of non-Newtonian fluids whose properties are described by the rheological equation

$$\tau_{ij} = -p\delta_{ij} + Fe_{ij},\ e_{ij} = \partial v_i/\partial x_j + \partial v_j/\partial x_i, \tag{3.1}$$

where τ_{ij} are components of the tensor of the viscous stresses (i, j = 1, 2, 3); δ_{ij} is the Kronecker symbol; e_{ij} are components of the strain-rate tensor; $v_1 \equiv u$, $v_2 \equiv v$, $v_3 = w$ are components of the velocity of the fluid in the Cartesian coordinate system $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$. We assume that F in (3.1) has the following structure:

$$F = \mu(T) f(I_2). \tag{3.2}$$

Here, f is an arbitrary function of the second (quadratic) invariant of the strain-rate tensor

$$I_2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2. \tag{3.3}$$

In particular, for the power-law fluid in (3.2) we need to put $f(I_2) = I_2^{(n-1)/2}$ (the value $n = 1$ corresponds to a Newtonian fluid); the Ellis model is characterized by the relation $f(I_2) = A + BI_2^{(n-1)/2}$, where A, B = const [14].

As before, we assume that the analog of viscosity – the consistency of the medium μ – is an exponential or power function of temperature.

Steady nonisothermal flows of non-Newtonian incompressible fluids which obey rheological law (3.1)-(3.2) are described by the equations

$$\rho v_i \frac{\partial v_j}{\partial x_i} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i}\left(F \frac{\partial v_j}{\partial x_i}\right) + \frac{\partial F}{\partial x_i}\frac{\partial v_i}{\partial x_j}; \tag{3.4}$$

$$\operatorname{div} \mathbf{v} = 0, \ \mathbf{v} = (v_1, v_2, v_3); \tag{3.5}$$

$$(\mathbf{v}\nabla) T = \varkappa \Delta T. \tag{3.6}$$

Here, we perform summation over the repeating subscript i; ρ is the density of the fluid. We seek the particular solutions of system (3.4)-(3.6) in the form

$$v_1 = 0, \ v_2 = 0, \ v_3 = w(x), \ p = p(x, z), \ T = T(x, z), \tag{3.7}$$

which corresponds to rectilinear flow in a plane channel. Then the inertial terms in the left side of equations of motion (3.4) are identically zero, while the continuity equation (3.5) is satisfied automatically. Considering this and using the notation $w_x' = dw/dx$, we find from (3.4)-(3.6) that

$$\partial p/\partial x = w_x' \partial F/\partial z; \tag{3.8}$$

$$\partial p/\partial z = (\partial/\partial x)(F w_x'); \tag{3.9}$$

$$w(x)\partial T/\partial z = \varkappa(\partial^2 T/\partial x^2 + \partial^2 T/\partial z^2). \tag{3.10}$$

Using cross differentiation to exclude pressure from (3.8)-(3.9), we obtain

$$(\partial^2/\partial z^2 - \partial^2/\partial x^2)(F w_x') = 0. \tag{3.11}$$

The general solution of Eq. (3.11) is written as

$$F w_x' = \Phi(z + x) + \Psi(z - x) \tag{3.12}$$

(Φ and Ψ are arbitrary functions).

It follows from Eqs. (3.2)-(3.3) and (3.7) that $F = \mu(T) f(w_x'^2)$. Considering this, we put $\Phi(\xi) = -Ae^{\alpha\xi}$, $\Psi(\xi) = Be^{\alpha\xi}$ in Eq. (3.12) (A, B, and α are arbitrary constants). As a result, we obtain the following equations for the velocity and temperature of the fluid

$$\mu(T) f\left(w_x'^2\right) w_x' = e^{\alpha z}\left(-Ae^{\alpha x} + Be^{-\alpha x}\right); \tag{3.13}$$

$$w(x)\partial T/\partial z = \varkappa(\partial^2 T/\partial x^2 + \partial^2 T/\partial z^2). \tag{3.14}$$

We will need to use these equations later.

4. Exact Solutions for an Exponential Temperature Dependence of the Consistency of the Medium. Let the consistency of the medium μ decrease exponentially with temperature in accordance with the law (1.1). We seek the particular solution of Eqs. (3.13)-(3.14) in the form

$$w = w(x), T = T_0 - \frac{\alpha}{\beta} z + \frac{\alpha}{\beta\varkappa} \theta(x), \tag{4.1}$$

which leads us to a system for the unknown functions w and θ:

$$f\left(w_x'^2\right) w_x' = \frac{1}{\mu_0} \exp\left(\frac{\alpha}{\varkappa}\theta\right) [-A\exp(\alpha x) + B\exp(-\alpha x)]; \tag{4.2}$$

$$w = -\theta_{xx}''. \tag{4.3}$$

Equations (4.2) and (4.3) were derived in [7] for a Newtonian fluid, which corresponds to $f \equiv 1$.

Excluding w from (4.2) and (4.3), we obtain a third-order nonlinear differential equation for the temperature component θ:

$$f(|\theta_{xxx}'''|^2)\theta_{xxx}''' = \frac{1}{\mu_0} \exp\left(\frac{\alpha}{\varkappa}\theta\right) [A\exp(\alpha x) - B\exp(-\alpha x)]. \tag{4.4}$$

For a power-law fluid, corresponding to $f(w_x'^2) = |w_x'|^{n-1}$, we write Eq. (4.4) as follows at $w_x' \leq 0$ after performing some elementary transformations

$$\theta_{xxx}''' = \exp\left(\frac{\alpha\theta}{\varkappa n}\right)\left[\frac{A}{\mu_0}\exp(\alpha x) - \frac{B}{\mu_0}\exp(-\alpha x)\right]^{1/n}. \tag{4.5}$$

When n = 1, it can be shown by direct proof that the function

$$\theta = \frac{\varkappa}{\alpha} \ln \frac{24\alpha^2 \varkappa \mu_0 AB}{(Ae^{\alpha x} + Be^{-\alpha x})^3} \tag{4.6}$$

is the exact (particular) solution of Eq. (4.5).

Now let us show that for any value of n, Eq. (4.5), with A = 0, B ≠ 0 or A ≠ 0, B = 0, can be reduced to the Blausius equation. The latter describes the hydrodynamic boundary layer on a flat plate.

In fact, at A ≠ 0, B = 0 (the case A = 0, B ≠ 0 is examined in a similar manner), if we make the substitution

$$\zeta = \frac{\alpha}{\varkappa n}\theta + \frac{\alpha}{n}x \tag{4.7}$$

we reduce Eq. (4.5) to an equation which is not explicitly dependent on x:

$$\zeta_{xxx}''' = -ae^{\zeta}, a = -\frac{\alpha}{\varkappa n}\left(\frac{A}{\mu_0}\right)^{1/n}. \tag{4.8}$$

We make the following substitution in (4.8)

$$g = -\zeta_x'. \tag{4.9}$$

As a result

$$g_{xx}'' = ae^{\zeta}. \tag{4.10}$$

Now differentiating both sides of this equation with respect to x, we have $g_{xxx}''' = ae^{\zeta}\zeta_x'$. Excluding the function ζ_x' and e^{ζ} from here by means of (4.9) and (4.10), we derive an equation for g:

$$g_{xxx}''' + gg_x'' = 0. \tag{4.11}$$

This equation is often encountered in the theory of hydrodynamic boundary layers [15] and, as (4.7), reduces to a first-order equation. It was established in [16] that there may be more than one solution of (4.11) (and, thus, of the initial equation (4.5)) for certain boundary conditions. Having the particular solution of Eq. (4.11), we can find the particular solution (4.8) at A ≠ 0, B = 0 from the formula

$$\theta = \frac{\varkappa n}{\alpha}\ln\left(\frac{g_{xx}''}{a}\right) - \varkappa x, \tag{4.12}$$

this solution being a consequence of Eqs. (4.7) and (4.10). For example, (4.11) permits the solution $g = 3(x + C)^{-1}$ (C being an arbitrary constant). With allowance for this, we can use (4.12) to find the particular solution of Eq. (4.5) at B = 0:

$$\theta = \frac{\varkappa n}{\alpha} \ln \left[\frac{6}{a(x+C)^3}\right] - \varkappa x.$$

We will use $g = g^{(B)}(x)$ to denote the Blausius solution satisfying Eq. (4.11) and the boundary conditions $g(0) = g_x'(0) = 0$, $g_x'(\infty) = 1$. This solution was tabulated in [15] for example. Using the group property of (4.11), it is easily shown that the two-parameter family of functions $g = \lambda g^{(B)}(\lambda x + \sigma)$ (where λ and σ are arbitrary constants) is the solution of (4.11). Inserting this relation into (4.12), we obtain a four-parameter family of solutions for the temperature profile.

5. Exact Solutions for a Power Dependence on the Consistency of the Medium on Temperature. Now let us examine a power law for the change in the consistency of the medium in relation to temperature (1.2). We seek particular solutions of Eqs. (3.13)-(3.14) in the form

$$w = w(x),\ T = T_0 e^{-\alpha z/m}\varphi(x). \tag{5.1}$$

As a result, to determine the functions w and φ we have the system:

$$\mu_0\varphi^{-m} f\left(w_x'^2\right) w_x' = -Ae^{\alpha x} + Be^{-\alpha x}; \tag{5.2}$$

$$-\frac{\alpha}{m} w\varphi = \varkappa\left(\varphi_{xx}'' + \frac{\alpha^2}{m^2}\varphi\right). \tag{5.3}$$

It follows from Eq. (5.3) that

$$w_x' = -\frac{\varkappa m}{\alpha}\left(\varphi_{xx}''/\varphi\right)_x'. \tag{5.4}$$

Excluding the derivative w_x' from (5.2), we can use (5.4) to derive a third-order equation for φ. For a Newtonian fluid, corresponding to $f \equiv 1$, this equation is written in the form

$$\left(\varphi_{xx}''/\varphi\right)_x' = \frac{\alpha}{m\varkappa\mu_0}\varphi^m\left(Ae^{\alpha x} - Be^{-\alpha x}\right). \tag{5.5}$$

It can be shown by direct proof that Eq. (5.5) has the particular solution

$$\varphi = \gamma(Ae^{\alpha x} + Be^{-\alpha x})^{-3/m},\ \gamma = [24AB\alpha^2\varkappa\mu_0(m+3)/m]^{1/m}. \tag{5.6}$$

For a power-law fluid, we obtain the following from (5.2), (5.4) at $w_x' \leq 0$

$$\left(\varphi_{xx}''/\varphi\right)_x' = \frac{\alpha}{\varkappa m}\varphi^{m/n}\left(\frac{A}{\mu_0}e^{\alpha x} - \frac{B}{\mu_0}e^{-\alpha x}\right)^{1/n}. \tag{5.7}$$

Now let us examine the flow in a narrow channel, which corresponds to $\alpha x \ll 1$. At A = B, (5.7) takes the form

$$\left(\varphi_{xx}''/\varphi\right)_x' = qx^{1/n}\varphi^{m/n},\ q = \frac{\alpha}{\varkappa m}\left(\frac{2\alpha A}{\mu_0}\right)^{1/n} \tag{5.8}$$

and permits the particular solution

$$\varphi = Cx^{\nu},\ \nu = -\frac{3n+1}{m},\ C = \left[\frac{q}{2\nu(\nu-1)}\right]^{n/(n-m)}.$$

At $A \neq B$, we obtain the following from (5.7) for $\alpha x \ll 1$

$$\left(\varphi_{xx}''/\varphi\right)_x' = \lambda\varphi^{m/n},\ \lambda = \frac{\alpha}{\varkappa m}\left(\frac{A-B}{\mu_0}\right)^{1/n}. \tag{5.9}$$

Changing over to the new variables

$$s = \varphi^2,\ u = (\varphi_x')^2, \tag{5.10}$$

we reduce the order of Eq. (5.9). As a result, we arrive at the Emden–Fowler equation $u_{ss}'' = (1/2)\lambda s^{(m-n)/2n} u^{-1/2}$. This equation is integrated in quadratures, for example, at n = m and m = 3n [17].

6. Certain Problems on Nonisothermal Flows in Plane Channels. Using the equations derived in Parts 4 and 5, we now formulate specific problems on nonisothermal flows in plane channels of the width 2h ($-h \le x \le +h$). For flows that are symmetrical with respect to the channel axis, when we allow for the condition of adhesion of the fluid to the wall we have three boundary conditions:

$$x = 0,\ \partial w/\partial x = 0;\ \ x = 0,\ \partial T/\partial x = 0;\ \ x = h,\ w = 0. \tag{6.1}$$

We also assume that the flow rate of the liquid Q is given. The quantity Q is integrally related to w:

$$Q = 2\int_0^h w(x)\,dx. \tag{6.2}$$

By having the transverse coordinate x approach zero in (3.13) and using the first condition of (6.1), we obtain the relationship between the coefficients A and B:

$$A = B. \tag{6.3}$$

Case 1. Let the temperature on the wall of the channel decrease linearly with the coordinate z by the law

$$x = h,\ T = T_0 - Ez, \tag{6.4}$$

while the consistency of the medium depends exponentially on temperature (1.1) (E is the temperature gradient along the walls).

We seek the velocity and temperature of the fluid in the form (4.1) at $\alpha = \beta E$, where w and θ satisfy system (4.2)-(4.3). Using boundary conditions (6.1), (6.4) and Eq. (6.3), after we exclude w we arrive at the following problem to determine the function θ:

$$f\left(|\theta'''_{xxx}|^2\right)\theta'''_{xxx} = \frac{2A}{\mu_0}\exp\left(\frac{\beta E}{\varkappa}\theta\right)\operatorname{sh}(\beta E x); \tag{6.5}$$

$$x = 0, \theta'_x = 0;\ x = h, \theta = 0;\ x = h, \theta''_{xx} = 0. \tag{6.6}$$

Here, the second condition of (6.6) is obtained by comparing (6.4) with Eqs. (4.1) at x = h. Boundary condition (6.4) is obtained by passing to the limit $x \to h$ in Eq. (4.3) with allowance for the condition of adhesion to the channel wall (6.1).

Equation (6.5) contains the unknown parameter A, which must be calculated in the following manner. We insert Eq. (6.2) into the right side. In accordance with (4.3), $w = -\theta_{xx}''$. After integrating, with allowance for the first boundary condition (6.5) we have

$$Q = -2\theta_x'(h). \tag{6.7}$$

This relation connects the sought parameter A with the specified flow rate Q. Having assigned A arbitrarily in (6.5) and having solved problem (6.5)-(6.6), we find Q from (6.7) as a function of A. Inverting this relation, we obtain A = A(Q).

For a Newtonian fluid ($f \equiv 1$), it follows from the results in [18] that problem (6.5)-(6.7) has two solutions for certain $0 < A < A_{max}$ and no solutions for $A > A_{max}$. The relation A(Q) at $\beta Eh \to 0$ (flow in a narrow channel) is shown in Fig. 2. The Poiseuille flow analog is obtained by passing to the limit $\alpha \to 0$ in (4.1)-(4.3), where it is necessary to put $A = B = C/\alpha$, C = const.

Case 2. Now let the temperature on the channel walls decrease exponentially with the coordinate z by the law

$$x = h,\ T = T_0 e^{-\lambda z}, \tag{6.8}$$

the temperature dependence of consistency obeying power law (1.2). We seek the solution in the form (5.1) with $\alpha = m\lambda$, where the functions w and φ are determined from Eqs. (5.2)-(5.3).

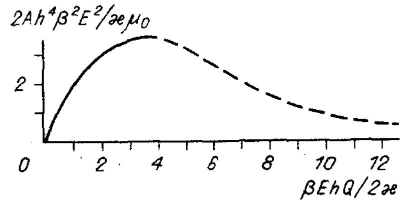

Fig. 2

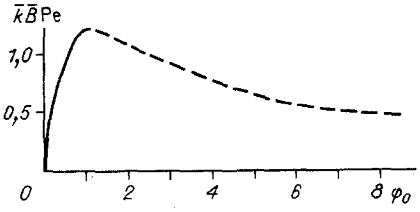

Fig. 3

Using boundary conditions (6.1), (6.8) and Eq. (6.3) and excluding w in the case of a power-law fluid, we obtain the problem for determining φ:

$$\left(\frac{\varphi''_{xx}}{\varphi}\right)'_x = \frac{\lambda}{\varkappa}\left(\frac{2A}{\mu_0}\right)^{1/n}\varphi^{m/n}\,[\mathrm{sh}\,(\lambda m x)]^{1/n}; \tag{6.9}$$

$$x = 0,\ \varphi'_x = 0;\ x = h,\ \varphi = 1;\ x = h,\ \varphi''_{xx} = -\lambda^2\varphi. \tag{6.10}$$

Here, the second boundary condition is found by comparing (6.8) with Eqs. (5.1) at x = h. The third condition of (6.10) is derived by passing to the limit $x \to h$ in (5.3) with allowance for the adhesion condition (6.1).

As before, we determine the free parameter A in (6.9) by means of (6.2) after using (5.3) to exclude w from the right side of the latter. As a result

$$Q = -\frac{2\varkappa}{\lambda}\int_0^h \frac{\varphi''_{xx}}{\varphi}\,dx - 2\varkappa\lambda h. \tag{6.11}$$

Having solved problem (6.9)-(6.10) with arbitrary A, we can use Eq. (6.11) to find the relation Q = Q(A). Inversion of this relation gives us the sought relation A = A(Q).

In accordance with this, we will show that for sufficiently large A Eqs. (6.9)-(6.10) have no solution. We make the substitutions $\bar{x} = x/h$, $\bar{\lambda} = \lambda m h$, $\bar{h}^2 = \lambda^2 h^2$, $\bar{\varphi} = \varphi - 1$, $\bar{A} = \frac{\lambda h^3}{\varkappa}\left(\frac{2A}{\mu_0}\right)^{1/n}$. Integrating (6.9) from $\bar{x}$ to 1, we reduce (6.9)-(6.10) to the integral equation

$$\bar{\varphi}(\bar{x}) = \bar{A}\int_0^1 G(\bar{x},\xi)\left\{(1+\bar{\varphi})\int_\xi^1 (1+\bar{\varphi})^{m/n}(\mathrm{sh}\,\bar{\lambda}t)^{1/n}dt\xi\, d\xi + \bar{h}^2\int_0^1 G(\bar{x},\xi)\bar{\varphi}(\xi)\,d\xi,\right.$$

where $G(\bar{x},\xi) = \begin{cases}1-\xi, & \bar{x}\leqslant\xi,\\ 1-\bar{x}, & \bar{x}\geqslant\xi\end{cases}$ is the Green's function of the operator $-\bar{\varphi}'' = 0$ with the boundary conditions $\bar{\varphi}(1) = 0$, $\bar{\varphi}'(0) = 0$. Since $\bar{\varphi}(\bar{x}) \geq 0$, $\bar{\varphi}''(\bar{x}) \leq 0$, then $\bar{\varphi}(\bar{x}) \geq \bar{\varphi}(0)(1-\bar{x}) = \bar{\varphi}_0(1-x)$. Considering this inequality, we arrive at the estimate

$$\bar{A} \leqslant \frac{\bar{\varphi}_0(2-\bar{h}^2)}{2\int_0^1 (1-\bar{x})\left\{(1+\bar{\varphi}_0(1-\bar{x}))\left[\int_{\bar{x}}^1 (1+\bar{\varphi}_0(1-t))^{m/n}(\mathrm{sh}\,\bar{\lambda}t)^{1/n}dt\right]\right\}d\bar{x}}.$$

It follows from this that the quantity A has an upper bound when $\bar{h}^2 < 2$, since $\bar{A} \to 0$ at $\bar{\varphi}_0 \to 0$ and $\bar{\varphi}_0 \to \infty$. Thus, the initial equation (6.9)-(6.10) has no solution at $\bar{A} > \bar{A}_{max}$. At $\bar{A} < \bar{A}_{max}$, each pressure gradient (shear stress on the wall) corresponds to two flow rates. It can be shown (see Part 2) that the branch of the solution on which the pressure gradient decreases with an increase in flow rate is unstable. Figure 3 shows the dependence of the pressure gradient on fluid temperature on the channel axis.

In the region of subcritical pressure gradients, system (5.2)-(5.3) can be solved by the small-parameter method. Let f = 1 (Newtonian fluid). We introduce the dimensionless quantities: $\omega = w2h/Q$, $\mathrm{Pe} = Q/2\kappa$, $\bar{k} = \alpha h/m$, $\bar{B} = A(\sinh m\bar{k})/\mu_0$, and we write system (5.2)-(5.3) as

$$\varphi'' + \bar{k}^2\varphi = -\mathrm{Pe}\,\bar{k}\omega\varphi,\quad \frac{d\omega}{dx} = -\bar{B}\varphi^m\frac{\mathrm{sh}\,(m\bar{k}x)}{\mathrm{sh}\,(m\bar{k})}. \tag{6.12}$$

In writing (6.12), we considered the symmetry of the flow (A = B). The parameter $\bar{B}$ is the shear stress on the channel wall, determined by the flow rate. We seek the solution of (6.12) in the form of series

$$\omega(\bar{x}) = \sum_{n=0}^{\infty} \bar{k}^n \omega_n,\ \varphi = \sum_{n=0}^{\infty} \bar{k}^n \varphi_n,\ \bar{B} = \sum_{n=0}^{\infty} \bar{k}^n \bar{B}_n.$$

For the zeroth approximation (isothermal Poiseuille flow) we have

$$d\omega_0/dx = -\bar{B}_0\bar{x},\ \varphi_0 = 1,\ \bar{B}_0 = 3,\ \omega_0 = (3/2)(1 - \bar{x}^2).$$

For the first approximation, we obtain the linear system

$$d\omega_1/dx = -3m\varphi_1 - \bar{B}_1\bar{x},\ \varphi_1'' = -\mathrm{Pe}\,\omega_0,$$

$$\varphi_1(1) = 0,\ \varphi_1'(0) = 0,\ \omega_1(1) = 0,\ \omega_1'(0) = 0,\ \int_0^1 \omega_1\, d\bar{x} = 0,$$

the solution of which we will represent as

$$\omega_1 = \frac{m\,\mathrm{Pe}}{560}\left[315\bar{x}^4 - 35\bar{x}^6 - 333\bar{x}^2 + 53\right],\ \varphi_1 = -\frac{\mathrm{Pe}}{8}\left[6\bar{x}^2 - \bar{x}^4 - 5\right],$$

$$\bar{B}_1 = -\frac{24}{35}m\,\mathrm{Pe}.$$

Now let us see how flows of the Newtonian fluid (n = 1) correspond to solution (5.6), which at A = B, $\alpha = \lambda m$ can be written in the form

$$\varphi = b\left[\mathrm{ch}(\lambda m x)\right]^{-3/m},\ b = \left[\frac{3\lambda^2\varkappa\mu_0 m(m+3)}{A}\right]^{1/m}. \tag{6.13}$$

Function (6.13) satisfies Eq. (6.9) and the first boundary condition of (6.10). Requiring that the second and third boundary conditions of (6.10) be satisfied along with Eq. (6.11), we obtain

$$\mathrm{ch}(\lambda m h) = \left[\frac{3\lambda^2\varkappa\mu_0 m(m+3)}{A}\right]^{1/3}; \tag{6.14}$$

$$\mathrm{th}^2(\lambda m h) = \frac{3m-1}{3(m+1)}; \tag{6.15}$$

$$Q = \frac{6\varkappa(m+3)}{m}\,\mathrm{th}(\lambda m h) - 20\varkappa\lambda h. \tag{6.16}$$

The first equation, (6.14), is used to determine the parameter A, while Eqs. (6.15)-(6.16) convey the character of the limitations on the flow parameters (λ, m, h, κ, and Q) and has physical significance at $m \geq 1/3$.

LITERATURE CITED

1. S. S. Kutateladze, Principles of the Theory of Heat Transfer, Atomizdat, Moscow (1979).
2. S. Bretschneider, Properties of Gases and Liquids (Engineering Methods of Calculation), Khimiya, Leningrad (1966).
3. J. I. Frenkel, Kinetic Theory of Liquids, Peter Smith, Gloucester, MA.
4. S. A. Rigirer, "Effect of heating on the viscous drag and steady-state unidimensional flow of a liquid in drop form," Prikl. Mat. Mekh., 22, No. 3 (1958).
5. S. A. Kaganov, "Steady-state laminar flow in a plane channel and a circular cylindrical pipe with allowance for frictional heat and the temperature dependence of viscosity," Zh. Prikl. Mekh. Tekh. Fiz., No. 3 (1962).
6. S. A. Bostandzhiyan, A. G. Merzhanov, and S. P. Khudyaev, "Hydrodynamic thermal explosion," Dokl. Akad. Nauk SSSR, 163, No. 1 (1965).
7. V. I. Naidenov, "Similarity of a problem of convective heat transfer," Zh. Prikl. Mekh. Tekh. Fiz., No. 5 (1974).
8. V. I. Naidenov and A. D. Polyanin, "Certain nonlinear convective effects in the theory of filtration and hydrodynamics," Dokl. Akad. Nauk SSSR, 279, No. 3 (1984).
9. P. Ya. Polubarinova, Theory of the Movement of Groundwater [in Russian], Nauka, Moscow (1977).
10. D. Joseph, Stability of Fluid Motions [Russian translation], Mir, Moscow (1981).
11. E. Yanke, F. Emde, and F. Lesch, Special Functions [Russian translation], Mir, Moscow (1981).

12. A. I. Wolpert and S. I. Khudaev, Analysis in Classes of Discontinuous Functions of Mathematical Physics [Russian translation], Nauka, Moscow (1975).
13. G. V. Keller, Some Position Problems in the Nonlinear Theory of Heat Generation. Theory of Branching and Nonlinear Eigenvalue Problems [Russian translation], Mir, Moscow (1974).
14. Z. P. Shul'man and B. M. Berkovskii, Boundary Layer of Non-Newtonian Fluids [in Russian], Nauka Tekh., Minsk (1966).
15. H. Schlichting, Boundary Layer Theory, 6th ed., McGraw-Hill, New York (1968).
16. P. Kazal', "Set of solutions of a boundary-layer equation," Mekhanika, No. 4 (1974).
17. V. F. Zaitsev and A. D. Polyanin, Discrete-Group Method of Integrating the Equations of Nonlinear Mechanics, Preprint, Inst. Probl. Mekh. Akad. Nauk SSSR, No. 339, Moscow (1988).
18. V. I. Naidenov, "Nonlinear equations of the self-similar nonisothermal motion of a viscous fluid," Zh. Vychisl. Mat. Mat. Fiz., 28, No. 12 (1988).

EFFECT OF A GAS CAVITY ON A PRESSURE SURGE IN A HYDRAULIC LINE

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Situations in which gas-filled cavities are present in the fluid are encountered in the operation of various hydraulic systems. This is sometimes the result of the accidental admission of air into the line, while in other cases it is due to the presence of air chambers placed in the system to damp pulsations of the fluid.

It is known that the presence of a macroscopic volume of gas in a hydraulic line can sometimes appreciably intensify pressure fluctuations occurring during transients [1-4]. For example, during the filling of a pipeline with fluid, a hydraulic shock 10 times greater than the pressure of the feed tank is realized [2]. The author of [3] studied the hydraulic shock which occurred when a pipeline provided with an air chamber and filled with a viscous fluid was rapidly connected to a tank under constant pressure. It was found that when the relative volume of air $\alpha_V < 10^{-2}$, the presence of a chamber designed to damp pressure surges leads to some increase in maximum pressure (by 30%). Only at $\alpha_V > 3 \cdot 10^{-2}$ does the chamber alleviate hydraulic shocks.

A numerical method was used in [4] to study the effect of the gas cavity on the pressure maximum for the case of instantaneous opening of a valve with a low hydraulic resistance. The investigation established the optimum gas volume at which the hydraulic-shock-induced increase in pressure would be maximal. This value is several times greater than the maximum pressure in a pipeline without a gas cavity. If the volume of the gas cavity is large enough, it acts as a damper and lowers the maximum pressure. Thus, depending on the parameters of the hydraulic system, a localized gas volume can either relieve pressure from a hydraulic shock or increase the pressure to a level which is dangerous for the system.

It should be noted that the authors of [1-4] did not study the effect of the loading of a pipeline by pressure. However, this parameter is important because a slow "application" of the load (gradual opening of a valve, etc.) is the method usually employed to eliminate dangerous pressure surges during transients in hydraulic systems.

In the present study, we experimentally and theoretically examine a transient involving the loading of a pipeline with pressure when the line has a gas cavity at the end. In contrast to [3, 4], the characteristic period of pressure build-up at the inlet of the system corresponded to several traversals of the line by a wave. Thus, the hydraulic-shock character of the transient was fairly weak.

A diagram of the test unit is shown in Fig. 1. One end of a steel pipe 5 with a length L = 2.3 m and a diameter d = 22 mm was connected by means of an adapter 2 and electromagnetic valve 1 to an air main at a pressure $P_1 = 7 \cdot 10^5$ Pa. A steel cylinder 6 with a

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